A.F. Diubuk: The problem of forecasting the pressure field from a full system of hydrodynamic equations.

(K zadache prognoza baricheskogo polia po polnoi sisteme uravnenii gidromekhaniki)

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THE PROBLEM OF FORECASTING THE PRESSURE FIELD FROM A FULL SYSTEM OF HYDROMECHANICS EQUATIONS   by A. F. Diubiuk

Certain generalizations are made of the solution of the problem of a short-range forecast made previously [1].

TR. by R.M. Holden

CALCULATION OF THE HORIZONTAL AND VERTICAL COMPONENTS OF THE CORIOLIS FORCE

The system of hydromechanics equations in a rectangular right system of coordinates is written as follows:

$$\begin{aligned}
&\frac{du}{dt} + 2\omega_y w - 2\omega_z v + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial \Phi}{\partial x} = 0,\\
&\frac{dv}{dt} + 2\omega_z u - 2\omega_x w + \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial \Phi}{\partial y} = 0, \qquad (1)\\
&\frac{dw}{dt} + 2\omega_x v - 2\omega_y u + \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\partial \Phi}{\partial z} = 0,
\end{aligned}$$

$$\frac{d\ln\rho}{dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \qquad (2)$$

$$\frac{d\ln T}{dt} - \frac{AR}{c_p}\frac{d\ln p}{dt} = 0, \qquad (3)$$

$$p = \rho RT, \qquad (4)$$

where Φ is the potential of Newtonian attraction, and the other symbols are as usual.

Let us select as the basic coordinate system that in which the x-axis is directed toward the east, the y-axis toward the north, and the z-axis vertically upward. Then,

$$\omega_x = 0, \quad \omega_y = 2\omega\cos\varphi, \quad \omega_z = 2\omega\sin\varphi,$$

whereupon

$$\begin{aligned}
&\frac{du}{dt} + 2\omega\cos\varphi w - 2\omega\sin\varphi v + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial \Phi}{\partial x} = 0;\\
&\frac{dv}{dt} + 2\omega\sin\varphi u + \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial \Phi}{\partial y} = 0, \qquad (1a)\\
&\frac{dw}{dt} - 2\omega\cos\varphi u + \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\partial \Phi}{\partial z} = 0.
\end{aligned}$$

This system will also be the initial system of equations.. Further, let us turn the x-axis such that it becomes parallel with the earth's axis. Here the old and new coordinates are connected by the formulas

$$\begin{aligned}
&x = x'; \quad y = y'\sin\varphi + z'\cos\varphi; \quad z = -y'\cos\varphi + z'\sin\varphi,\\
&x' = x; \quad y' = y\sin\varphi - z\cos\varphi; \quad z' = x\cos\varphi + z\sin\varphi.
\end{aligned} \qquad (5)$$

From this,

$$u = u', \quad v = v'\sin\varphi + w\cos\varphi, \quad w = -v\cos\varphi + w\sin\varphi,$$
$$\frac{du}{dt} = \frac{du'}{dt}, \quad \frac{dv}{dt} = \frac{dv'}{dt}\sin\varphi + \frac{dw}{dt}\cos\varphi, \quad \frac{dw}{dt} = -\frac{dv'}{dt}\cos\varphi + \frac{dw'}{dt}\sin\varphi,$$
$$u' = u, \quad v' = v\sin\varphi - w\cos\varphi, \quad w' = v\cos\varphi + w\sin\varphi, \tag{6}$$
$$\frac{du'}{dt} = \frac{du}{dt}, \quad \frac{dv'}{dt} = \frac{dv}{dt}\sin\varphi - \frac{dw}{dt}\cos\varphi, \quad \frac{dw'}{dt} = \frac{dv}{dt}\cos\varphi + \frac{dw}{dt}\sin\varphi.$$

Now at the same time, $\omega_{x'} = \omega_{y'} = 0,\ \omega_{z'} = \omega$, i.e., according to (1),

$$\frac{du'}{dt} - 2\omega v' + \frac{1}{\rho}\frac{\partial p}{\partial x'} + \frac{\partial \Phi}{\partial x'} = 0,$$
$$\frac{dv'}{dt} + 2\omega u' + \frac{1}{\rho}\frac{\partial p}{\partial y'} + \frac{\partial \Phi}{\partial y'} = 0, \tag{1б}$$
$$\frac{dw'}{dt} + \frac{1}{\rho}\frac{\partial p}{\partial z'} + \frac{\partial \Phi}{\partial z'} = 0.$$

Setting, as in (1)

$$Q = RT_0 \ln\frac{p}{p_0} + \Phi, \quad c^2 = \frac{c_p}{c_v}RT_0, \quad \vartheta = \frac{T - T_0}{T_0},$$

we have

$$u'_t - 2\omega v' + Q_{x'} = -[\vartheta(Q_{x'} - \Phi_{x'}) + u'u'_x + v'u_y + w'u'_z] \equiv F'_1,$$
$$v'_t + 2\omega u' + Q_{y'} = -[\vartheta(Q_{y'} - \Phi_{y'}) + u'v'_x + v'v'_y + w'v'_z] \equiv F'_2,$$
$$w'_t + Q_{z'} = -[\vartheta(Q_{z'} - \Phi_{z'}) + u'w'_z + v'w'_y + w'w'_z] \equiv F'_3, \tag{7}$$
$$\frac{1}{c^2}Q_t + u'_{x'} + v'_{y'} + w'_{z'} =$$
$$= -\frac{1}{c^2}[u'(Q_{x'} - \Phi_{x'}) + v'(Q_{y'} - \Phi_{y'}) + w'(Q_{z'} - \Phi_{z'})] \equiv F'_4$$

Considering here that the magnitude $1/c^2$ is negligibly slight, and excluding u', v', and w' from (7), we get for Q the equation

$$\begin{vmatrix} \frac{\partial}{\partial t}, & -2\omega, & 0, & \frac{\partial}{\partial x} \\ 2\omega, & \frac{\partial}{\partial t}, & 0, & \frac{\partial}{\partial y'} \\ 0 & 0, & \frac{\partial}{\partial t}, & \frac{\partial}{\partial z'} \\ \frac{\partial}{\partial x'}, & \frac{\partial}{\partial y'}, & \frac{\partial}{\partial z'}, & 0 \end{vmatrix} Q = \begin{vmatrix} \frac{\partial}{\partial t}, & -2\omega, & 0, & F'_1 \\ 2\omega, & \frac{\partial}{\partial t}, & 0, & F'_2 \\ 0 & 0, & \frac{\partial}{\partial t}, & F'_3 \\ \frac{\partial}{\partial x'}, & \frac{\partial}{\partial y'}, & \frac{\partial}{\partial z'}, & F'_4 \end{vmatrix} \equiv F^{(Q)}. \tag{8}$$

Opening the determinant on the left side we have

$$\left(\frac{\partial^2}{\partial t^2}\Delta_1 + 4\omega^2\frac{\partial^2}{\partial z'^2}\right)Q = -F^{(Q)}, \tag{9}$$

where

$$\Delta_1 = \frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2}.$$

The solution of (9) for limitless space has the form [1]

$$Q = -\frac{1}{4\pi}\iiint_{-\infty}^{+\infty}\left[\Delta_1\overset{0}{Q}_1\frac{1}{R'_1}J_0\left(\frac{2\omega r' t}{R'_1}\right) + \Delta_1\overset{0}{Q}_t\frac{1}{R'_1}\int_0^t J_0\left(\frac{2\omega r'\tau}{R'}\right)d\tau\right]d\bar{x}'\,d\bar{y}'\,d\bar{z}' +$$
$$+ \iiiint_{-\infty\ 0}^{\infty\ t}\left[-(F'_{1x'} + F'_{2y'} + F'_{3z'})\frac{1}{R'_1}\frac{\partial}{\partial t}J_0\left(\frac{2\omega r'(t-\tau)}{R'_1}\right) + 2\omega(F'_{1y'} - F'_{2x'}) \times\right.$$

$$\times \frac{1}{R_1'} J_0 \left(\frac{2\omega r'(t-\tau)}{R_1'} - \frac{4\omega^2}{R_1'} F_{3z'}' \int_0^{t-\tau} J_0\left(\frac{2\omega r'\tau'}{R_1'}\right) d\tau'\right] d\tau\, d\bar{x}'\, d\bar{y}'\, d\bar{z}'. \qquad (10)$$

Turning to the initial coordinate system from formulas (5) and (6), let us note that due to the orthogonality of the coordinate transformation

$$\Delta_1 = \Delta, \quad R_1' = R_1, \quad d\bar{x}'\, d\bar{y}'\, d\bar{z}' = d\bar{x}\, d\bar{y}\, d\bar{z},$$

however,

$$r'^2 = R'^2 - (z' - \bar{z}')^2 = (x' - \bar{x}')^2 + (y' - \bar{y}')^2 \sin\varphi + (z' - \bar{z}')^2 \cos^2\varphi \equiv S_1^2,$$

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x}; \quad \frac{\partial}{\partial y'} = \sin\varphi \frac{\partial}{\partial y} - \cos\varphi \frac{\partial}{\partial z}; \quad \frac{\partial}{\partial z'} = \cos\varphi \frac{\partial}{\partial y} + \sin\varphi \frac{\partial}{\partial z},$$

$$F_1' = F_1; \quad F_2' = F_2 \sin\varphi - F_3 \cos\varphi; \quad F_3' = F_2 \cos\varphi + F_3 \sin\varphi,$$

where F_1, F_2 and F_3 are equal, respectively, to the right sides of equations (7) with the elimination of primes on all letters.

Then,

$$Q = -\frac{1}{4\pi} \iiint_{-\infty}^{+\infty} \left[\Delta \overset{0}{Q} \frac{1}{R_1} J_0\left(\frac{2\omega S_1}{R_1} t\right) + \Delta \overset{0}{Q}_t \frac{1}{R_1} \int_0^t J_0\left(\frac{2\omega S_1}{R_1} \tau\right) d\tau\right] dx'\, dy'\, dz' +$$
$$+ \iiint_{-\infty}^{\infty} \int_0^t \Big\{[-(F_{1x} + F_{2y} + F_{3z})] \frac{1}{R_1} \frac{\partial}{\partial t} J_0\left(\frac{2\omega S_1}{R_1}(t-\tau)\right) +$$
$$+ 2\omega [\sin\varphi (F_{1y} - F_{2x}) - \cos\varphi (F_{1z} - F_{3z})] \frac{1}{R_1} J_0\left(\frac{2\omega S_1}{R_1}(t-\tau)\right) -$$
$$- \frac{4\omega^2}{R_1} [F_{2y} \cos^2\varphi + (F_{2z} - F_{3y}) \sin\varphi \cos\varphi + F_{3z} \sin^2\varphi] \times$$
$$\times \int_0^{t-\tau} J_0\left(\frac{2\omega S_1}{R_1} \tau'\right) d\tau'\Big\} d\tau\, dx'\, dy'\, dz'. \qquad (11)$$

When solving the problem for a half-spcae we will consider that when $z = 0$, $Q_z = 0$. Separating in (11) the integrals which are functions of z and z', we can represent them thus:

$$\int_{-\infty}^{\infty} \varphi(z') \psi[(z-z')^2]\, dz' = \int_0^{\infty} \{\varphi(z') \psi[(z-z')^2] + \varphi(-z') \psi[(z+z')^2]\}\, dz',$$

requiring that

$$\frac{\partial}{\partial z} \int_0^{\infty} \{\varphi(z') \psi[(z-z')^2] + \varphi(-z') \psi[(z+z')^2]\}_{z=0}\, dz' =$$
$$= \int_0^{\infty} \{\varphi(z') \psi'(z'^2)\, 2(-z') + \varphi(-z') \psi'(z'^2)\, 2z'\}\, dz' = 0,$$

we have $-\varphi(z') + \varphi(-z') = 0$, i.e.,

$$\varphi(-z') = \varphi(z').$$

This indicates that for a half-space in formula (11) we should consider the integrals of the examined type equal to

$$\int_{-\infty}^{\infty} \varphi(z') \psi[(z-z')^2]\, dz' = \int_0^{\infty} \varphi(z') \{\psi[(z-z')^2] + \psi[(z+z')^2]\}\, dz'.$$

In other words, setting

$$R_1^2 = (x-x')^2 + (y-y')^2 + (z-z')^2,$$
$$R_2^2 = (x-x')^2 + (y-y')^2 + (z+z')^2,$$
$$S_1^2 = (x-x')^2 + (y-y')^2 \sin^2\varphi + (z-z')\cos^2\varphi,$$
$$S_2^2 = (x-x')^2 + (y-y')^2 \sin^2\varphi + (z+z')^2\cos^2\varphi,$$

we find that

$$Q = -\frac{1}{4\pi}\int_{-\infty}^{+\infty}\int\int_{0}^{\infty}\Delta\overset{0}{Q}\left(\frac{J_0\left(\frac{2\omega t S_1}{R_1}\right)}{R_1}+\frac{J_0\left(\frac{2\omega t S_2}{R_2}\right)}{R_2}\right)dz'\,dx'\,dy' + \ldots \quad (12)$$

The obtained integral-differential equation, as in (1), can be solve d numerically.

In the solution of (11) the Laplacian of Q_t^o enters into the right side. Direct practical use of this value presents considerable difficulties. As an approximate value of this magnitude we can use the value of Q_t obtained by interpolation when t = 0, allowing that with small t's we can consider $Q(t) = a_o + a_1 t + a_2(t^2/2)$, where the coefficients of a_1 are found from the following conditions:

$$\text{when } t = 0 \quad Q = \overset{0}{Q},$$
$$\text{when } t = -\delta t \quad Q = \overset{-1}{Q},$$
$$\text{when } t = -2\delta t \quad Q = \overset{-2}{Q},$$

whence

$$a_0 = \overset{0}{Q};\quad a_1 = \frac{4(\overset{0}{Q}-\overset{-1}{Q})+\overset{-2}{Q}}{2\delta t};\quad a_2 = \frac{\overset{0}{Q}-2\overset{-1}{Q}+\overset{-2}{Q}}{\delta t^2}.$$

Thus,

$$\overset{0}{Q}_t = \frac{4(\overset{0}{Q}-\overset{-1}{Q})+\overset{-2}{Q}}{2\delta t},$$

i.e., Q_t is determined from Q values at moments $T \neq 0, -\delta t, -2\delta t$.

THE RECURRENT SYSTEM FOR SOLVING THE PROBLEM OF FORECASTING WITH CONSIDERATION OF INTERNAL FRICTION

Let us examine the system of equations

$$\frac{du}{dt} + 2\omega\cos\varphi w - 2\omega\sin\varphi v - \nu\Delta u + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial\Phi}{\partial x} = 0,$$
$$\frac{dv}{dt} + 2\omega\sin\varphi u - \nu\Delta v + \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{\partial\Phi}{\partial y} = 0,$$
$$\frac{dw}{dt} - 2\omega\cos\varphi u - \nu\Delta w + \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\partial\Phi}{\partial z} = 0, \quad (13)$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Substitution of variables according to the formulas

$$x = x', \quad y = y'\sin\varphi + z'\cos\varphi, \quad z = -y'\cos\varphi + z'\sin\varphi$$

leads to the system

$$\begin{aligned} &u'_t - 2\omega v' - \nu\Delta u' + Q_{x'} = F'_1, \\ &v'_t + 2\omega u' - \nu\Delta v' + Q_{y'} = F'_2, \\ &w'_t - \nu\Delta w' + Q_{z'} = F'_3, \\ &u'_{x'} + v'_{y'} + w'_{z'} = 0. \end{aligned} \tag{14}$$

Excluding velocity components we have

$$\left[\left(\frac{\partial}{\partial t} - \nu\Delta_1\right)^2 \Delta_1 + 4\omega^2 \frac{\partial^2}{\partial z'^2}\right] Q = -F^{(Q)}. \tag{15}$$

Assuming in dimensionless values that ν is small, we seek a solution in the form

$$Q = q_0 + \nu q_1 + \nu^2 q_2 + \ldots, \tag{16}$$

which gives the recurrent system

$$\begin{aligned} &\left(\frac{\partial^2}{\partial t^2}\Delta_1 + 4\omega^2 \frac{\partial^2}{\partial z'^2}\right) q_0 = F^{(Q)}, \\ &\left(\qquad\qquad\right) q_1 = 2\frac{\partial}{\partial t}\Delta^2 q_0, \\ &\left(\qquad\qquad\right) q_2 = 2\frac{\partial}{\partial t}\Delta^2 q_1 - \Delta^3 q_0, \\ &\left(\qquad\qquad\right) q_3 = 2\frac{\partial}{\partial t}\Delta^2 q_2 - \Delta^3 q_1, \\ &\ldots\ldots\ldots \\ &\left(\qquad\qquad\right) q_{n+1} = 2\frac{\partial}{\partial t}\Delta^3 q_n - \Delta^2 q_{n-1}. \end{aligned} \tag{17}$$

We have solved these equations in the preceding section.

THE SOLUTION IN A SPHERICAL COORDINATE SYSTEM UNDER LOCAL CONDITIONS

In an immobile system of Cartesian coordinates with the coordinate origin at the center of the earth with the z-axis directed northward along the earth's axis,* the motion equations have the form

$$\begin{aligned} \frac{du}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} - \frac{\partial \Phi}{\partial x}, \\ \frac{dv}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} - \frac{\partial \Phi}{\partial y}, \\ \frac{dw}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - \frac{\partial \Phi}{\partial z}. \end{aligned} \tag{18}$$

It is easy to obtain a system of motion equations in the same coordinates as before, but confined to the earth

$$\begin{aligned} \frac{du'}{dt} + 2\omega v' &= -\frac{1}{\rho}\frac{\partial p}{\partial x_1} - \frac{\partial}{\partial x_1}\left(\Phi - \omega^2\frac{x_1^2 + y_1^2}{2}\right), \\ \frac{dv'}{dt} - 2\omega u' &= -\frac{1}{\rho}\frac{\partial p}{\partial y_1} - \frac{\partial}{\partial y_1}\left(\Phi - \omega^2\frac{x_1^2 + y_1^2}{2}\right), \\ \frac{dw'}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial z_1} - \frac{\partial}{\partial z_1}\left(\Phi - \omega^2\frac{x_1^2 + y_1^2}{2}\right), \end{aligned} \tag{19}$$

where ω is the angular valocity of the earth's rotation.

The formulas for changing from one coordinate system to another have the form

$$\begin{aligned} x &= x_1 \cos \omega t - y_1 \sin \omega t, \\ y &= x_1 \sin \omega t + y_1 \cos \omega t, \\ z &= z_1, \end{aligned} \tag{20}$$

$$\begin{aligned} x_1 &= x \cos \omega t + y \sin \omega t, \\ y_1 &= - x \sin \omega t + y \sin \omega t, \\ z_1 &= z. \end{aligned} \tag{20a}$$

If we convert from equations (1) to a rotary spherical system of coordinates (r, φ, λ), where r is the radius vector, φ is latitude and λ is longitude, we get, as is known,

$$\frac{dv_r}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + 2\omega v_\lambda \cos\varphi + \frac{v_\varphi^2 + v_\lambda^2}{r} - \frac{\partial}{\partial r}\left(\Phi - \frac{\omega^2 r^2}{2}\cos^2\varphi\right), \tag{21}$$

$$\frac{dv_\varphi}{dt} = -\frac{1}{\rho r}\frac{\partial p}{\partial \varphi} - 2\omega v_\lambda \sin\varphi - 2\frac{v_r v_\varphi}{r} - \frac{v_\lambda^2}{r}\operatorname{tg}\varphi - \frac{1}{r}\frac{\partial}{\partial \varphi}\left(\Phi - \frac{\omega^2 r^2}{2}\cos^2\varphi\right),$$

$$\frac{dv_\lambda}{dt} = -\frac{1}{\rho r\cos\varphi}\frac{\partial p}{\partial \lambda} - 2\omega v_r \cos\varphi + 2\omega v_\varphi \sin\varphi - 2\frac{v_r v_\lambda}{r} - 2\frac{v_\lambda v_\varphi}{2}\operatorname{tg}\varphi - \frac{1}{r\cos\varphi}\frac{\partial}{\partial \lambda}\left(\Phi - \frac{\omega^2 r^2}{2}\cos^2\varphi\right).$$

The formulas for the transition from an immobile system of coordinates to a rotary spherical system have the form

$$\begin{aligned} x &= r\cos\varphi\cos(\lambda - \omega t), \\ y &= -r\cos\varphi\sin(\lambda - \omega t), \\ z &= r\sin\varphi. \end{aligned} \tag{22}$$

Using (3) and (5) let us find the formulas for the transition from a rotary Cartesian coordinate system to a rotary spherical system:

$$\begin{aligned} x_1 &= r\cos\varphi\cos\lambda, \\ y_1 &= -r\cos\varphi\sin\lambda, \\ z_1 &= r\sin\varphi. \end{aligned} \tag{23}$$

Relationships (23) make it possible to convert from system (19) to system (21) and vice versa. Let us note that the relationships

$$\begin{aligned} u_1 &= v_r\cos\varphi\cos\lambda - v_\varphi\sin\varphi\cos\lambda - v_\lambda\sin\lambda, \\ v_1 &= -v_r\cos\varphi\sin\lambda - v_\varphi\sin\varphi\sin\lambda - v_\lambda\cos\lambda, \\ w_1 &= v_r\sin\varphi + v_\varphi\cos\varphi, \end{aligned} \tag{24}$$

$$\begin{aligned} v_r &= u_1\cos\varphi\cos\lambda - v_1\cos\varphi\sin\lambda + w_1\sin\varphi, \\ v_\varphi &= -u_1\sin\varphi\cos\lambda + v_1\sin\varphi\sin\lambda + w_1\cos\varphi, \\ v_\lambda &= -{}_1u\sin\lambda - v_1\cos\lambda. \end{aligned} \tag{24a}$$

occur between the velocities in these two systems. Thus we will assume that we are given a system of differential equations (21) in a spherical coordinate system. Using formulas (20a) and (24a) let us convert it to system (19) in rectangular coordinates.

Supplementing it with a continuity equation and designating, as before,

$$Q = RT_0 \ln \frac{p}{p_0} + \Phi, \; \vartheta = \frac{T - T_0}{T_0}, \; c^2 = \frac{c_p}{c_v} RT_0,$$

we reach a system analogous to (7) and, further, we get equation (9) and solution (10)..

After this in the obtained solution it is necessary to return to the spherical coordinate system, using formulas (23), and for velocities entering in F_1 (and derivatives of F_1), using solution (10), formula (24).

In formula (10) we should now replace $\int_{-\infty}^{\infty}\int\int \ldots d\overline{x}' \, d\overline{y}' \, d\overline{z}'$ by $\int_{-\pi/2}^{+\pi/2}\int_0^{2\pi}\int_0^{\infty} \ldots r^2 \cos\varphi \, dr \, d\lambda \, d\varphi$. on the right side of the integral, Further, we have

$$\frac{\partial}{\partial x_1} = \cos\varphi\cos\lambda \frac{\partial}{\partial r} - \frac{\sin\varphi\cos\lambda}{r}\frac{\partial}{\partial\varphi} - \frac{\sin\lambda}{r\cos\varphi}\frac{\partial}{\partial\lambda},$$

$$\frac{\partial}{\partial y_1} = -\cos\varphi\sin\lambda \frac{\partial}{\partial r} + \frac{\sin\varphi\sin\lambda}{r}\frac{\partial}{\partial\varphi} - \frac{\cos\lambda}{r\cos\varphi}\frac{\partial}{\partial\lambda},$$

$$\frac{\partial}{\partial z_1} = \sin\varphi \frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\frac{\partial}{\partial\varphi},$$

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\cos^2\varphi}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\lambda^2} + \frac{\operatorname{ctg}\lambda}{r^2}\frac{\partial}{\partial\lambda},$$

$$R_1^2 = (x_1 - x_1')^2 + (y_1 - y_1')^2 + (z_1 - z_1')^2 = r^2 + r'^2 - 2rr'\cos\gamma,$$

$$r_1^2 = (x_1 - x_1')^2 + (y_1 - y_1')^2 = r^2\cos^2\varphi - 2rr'\cos\varphi\cos(\lambda - \lambda') + r'^2\cos^2\varphi,$$

$$\cos\gamma = \cos\varphi\cos\lambda\cos\varphi'\cos\lambda' + \cos\varphi\sin\lambda\cos\varphi'\sin\lambda' + \sin\varphi\sin\varphi'.$$

Let us now separate from the third integral that integral with respect to r. For brevity let us examine, e.g., only the integral M (the others are analogous):

$$M = \int_0^{\infty} \Delta \overset{0}{Q} \frac{1}{R_1} J_0\left(\frac{2\omega r_1 t}{R_1}\right) r'^2 \, dr'.$$

We find that M has the form (a is the earth's radius):

$$M = \int_0^{\infty} \psi(r') f(r^2, r'^2, rr') \, dr = \left(\int_0^a + \int_a^{\infty}\right) \psi(r') f(r^2, r'^2, rr') \, dr' =$$

$$= \left(\int_a^{\infty} - \int_a^0\right) \psi(r') f^0(r^2, r'^2, rr') \, dr'.$$

Further, setting $r' = a^2/r''$ in the integral from a to 0, we get

$$M = \int_a^{\infty} \psi(r') f(r^2, r'^2, rr') \, dr' + \int_a^{\infty} \psi\left(\frac{a^2}{r''}\right) f_1\left(r^2, \frac{a^4}{r''^2}, \frac{a^2 r}{r''}\right) \frac{a^2 dr''}{r''^2} =$$

$$= \int_a^{\infty} \left[\psi(r') f(r^2, r'^2, rr') + \psi\left(\frac{a^2}{r'}\right) f_1\left(r^2, \frac{a^4}{r'^2}, \frac{a^2 r}{r'}\right)\frac{a^2}{r'^2}\right] dr'.$$

Let us require that when $r = a$, $\partial Q/\partial r = 0$. For this it suffices to set

$$\left[\psi(r')\frac{\partial f}{\partial r} + \psi\left(\frac{a^2}{r'}\right)\frac{\partial f_1}{\partial r_1}\frac{a^2}{r'^2}\right]_{r=a} = 0,$$

or

$$\psi\left(\frac{a^2}{r'}\right) = -\psi(r')\left(\frac{\frac{\partial f}{\partial r}}{\frac{\partial f_1}{\partial r}}\right)_{r=a}\left(\frac{r'}{a}\right)^2 .$$

This indicates that

$$M = \int_a^\infty \psi(z')\left[f(r^2, r'^2, rr') - \left(\frac{\frac{\partial f}{\partial r}}{\frac{\partial f_1}{\partial r}}\right)_{r=a} f_1\left(r^2, \frac{a^4}{r'^2}, \frac{a^2}{r'}\right)\right]dr'.$$

Analogous transformations are made for all integrals with respect to r'; however, since $\left(\frac{\frac{\partial f}{\partial r}}{\frac{\partial f_1}{\partial r}}\right)_{r=a}$ is a function not only of φ' and λ' but of φ and λ, the solution of this hold only for the local problem, when φ and λ in this factor can be considered equal to φ_0 and λ_0, i.e., constant.

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LITERATURE CITED

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